Question: Determine, through investigation, the equations of the lines that have a slope of 2 and that intersect the quadratic function $f(x) = x(x-6)$ once, twice, never.

All lines with a slope of 2 have the equation:

$$g(x) = 2x + k$$

where k is the y intercept. Changing the value of k shifts the line vertically so that it can intersect the parabola in one of three ways, as shown on the graph.

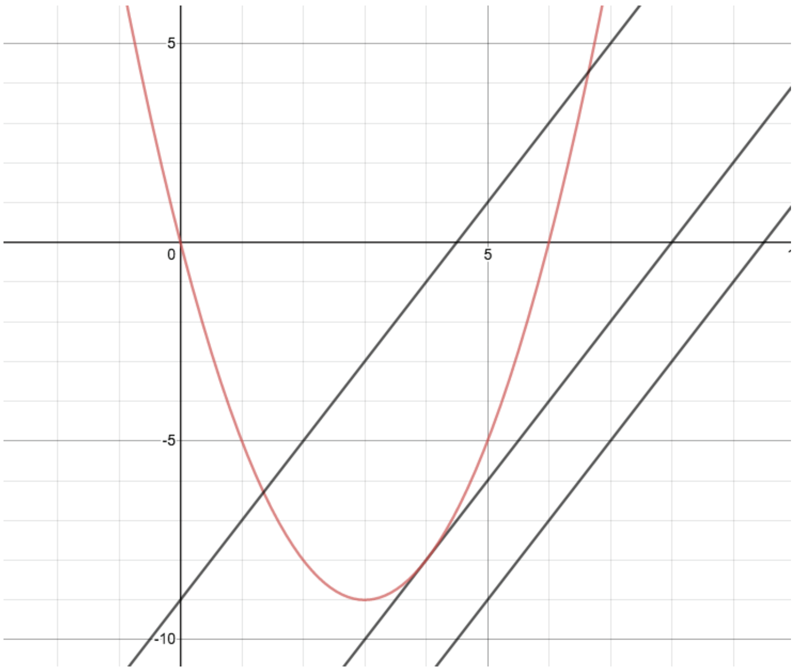

Figure 1: Intersection of parabola and lines with slope of 2.

Since we are interested in how many intersection points there are, we will use the discriminant:

$$b^2 - 4ac$$

where a, b and c are the coefficients in the quadratic standard form $ax^2 + bx + c$.

There are two main steps to answer this question. First equate the two right sides of the equations and solve to find x values of the points of intersection—even though there may be none. This is what the unknown value of k tells us. Second, apply the discriminant to find appropriate values of k.

Step 1:

$$x(x-6) = 2x + k$$

Since we are solving a quadratic equation, we need "standard form equals zero." Therefore, expand the LHS, move all terms to the left and simplify to standard form. This is the result:

$$x^2 - 8x - k = 0$$

The three coefficients are:

$$a = 1,\ b = -8,\ c = -k$$

The discriminant is:

$$\begin{aligned} & b^2 - 4ac \\ &= (-8)^2 - 4(1)(-k) \\ &= 64 + 4k \\ &= 4(16 + k) \end{aligned}$$

Case 1: The discriminant is less than zero and there is no point of intersection.

$$4(16 + k) < 0$$

$$16 + k < 0$$

$$k < -16$$

Therefore when $k < -16$ the y intercept is less than -16 and the line intersects with the parabola at no points.

Case 2: The discriminant is equal to zero and there is one solution to the quadratic equation.

$$4(16 + k) = 0$$

$16 + k = 0$

$k = -16$

Therefore when $k = -16$ the y intercept is -16 and the line intersects with the parabola at one point.

Case 3: The discriminant is greater than zero and there are two solutions to the quadratic equation.

$4(16 + k) > 0$

$16 + k > 0$

$k > -16$

Therefore when $k > 16$ the y intercept is greater than -16 and the line intersects with the parabola at two points.